Solitons in isotropic antiferromagnets: beyond a sigma model.

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Isotropic antiferromagnets shows a reach variety of magnetic solitons with non-trivial static and dynamic properties. One-dimensional soliton elementary excitations have a periodic dispersion law. For two-dimensional case, planar antiferromagnetic vortices having non-singular macroscopic core with the saturated magnetic moment are present. The dynamic properties of these planar antiferromagnetic vortex are characterized by presence of a gyroforce

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I. INTRODUCTION

Magnetically ordered materials (magnets) are known as essentially nonlinear systems and shows a large variety of localized nonlinear excitations with finite energy, or solitons, see Refs. 1,2,3,4. It is sufficient to note kink-type solitons (domain walls) which destroy long range order in one- dimensional systems; magnetic vortices, which cause a Berezinskii-Kosterletz-Thouless transition in two-dimensional magnets with continuous degeneration;[5,6] and also two-dimensional localized solitons like Belavin-Polyakov solitons,[7] see for review Refs. 4. All these solitons were firstly introduced in physics of magnets, and the development of soliton concept for this particular region of physics is believed to be important for modern nonlinear general physics of condensed matter as well as for field models of high-energy physics, see Ref. 8.

To date, solitons in Heisenberg ferromagnets, whose dynamics are described by the Landau–Lifshitz equation for the constant-length magnetization vector, have been studied in details. From a microscopic point of view description of such magnets is based on the Landau-Lifshitz equation for a unit (normalized) magnetization vector $\mathbf{m}$, $\mathbf{m}^2 = 1$, see Refs. 1,2,4. Basically, for antiferromagnets one can use a set of two equations for magnetizations of sublattices, which are unit vectors $\mathbf{m}_1$ and $\mathbf{m}_2$, or, that is more convenient, their irreducible combinations

$$\mathbf{m} = (\mathbf{m}_1 + \mathbf{m}_2)/2,\ \mathbf{l} = (\mathbf{m}_1 - \mathbf{m}_2)/2, \tag{1}$$

which are bound by constraint

$$(\mathbf{m},\ \mathbf{l}) = 0,\ \mathbf{m}^2 + \mathbf{l}^2 = 1. \tag{2}$$

These variables naturally reflect the symmetry inherent to antiferromagnets, regarding sublattices rearrangement and they are convenient for presentation of phenomenological energy of antiferromagnet. However, the growing of the number of variables essentially complicates the analysis, and within the framework of this approach a few works have been done, we point out Refs. 9,10,11.

A considerable progress in study of non-linear dynamics of antiferromagnets has been reached after obtaining of so-called $\sigma-$model, which presents a dynamical equation for the antiferromagnet vector $\mathbf{l}$ see for review Refs. 1,2,3,4,12,13,14,15. While deducing the model one considers that $\mathbf{m}$ is small, $\mathbf{m}^2 \ll 1$, where $\mathbf{m}$ is a slave variable and is determined by the vector $\mathbf{l}$ and its time derivative $\partial\mathbf{l}/\partial t$. $\sigma-$model equations can be derived either directly from the Landau-Lifshitz equations for sublattices magnetizations,[16,17] or phenomenologically, by account taken of symmetry considerations.[18] It is a common belief that description of nonlinear dynamics of antiferromagnets within $\sigma-$model has the same level of universality as within the Landau-Lifshitz equations for ferromagnets. At least it is considered to be true for low-frequent dynamics in the longwave approximation.

It is worth noting, the transition to σ-model is not connected with any expansion over small amplitudes of deviations of the vector $\mathbf{l}$ from the equilibrium position. Hence, $\sigma-$model is highly nonlinear. Since within $\sigma-$model $\mathbf{l}$ is considered as a unit vector, this model is a typical nonlinear chiral model, in which a non-linearity is determined a geometric condition $\mathbf{l}^2 = 1$. However, it turns out that an isotropic $\sigma-$model as a nonlinear system is to a certain extent quite “poor”. In particular, for a non-localized nonlinear wave of a structure $l_x + il_y = l_0 \cdot \exp(\mathbf{kr} - \omega t)$, $l_z = \sqrt{1 - l_0^2} = \text{const}$, where the wave amplitude $l_0 < 1$ can be not small, the frequency ω for a given “wave vector” $\mathbf{k}$ is independent on the wave amplitude l_0. As well, in this system there are no traveling-wave solitons, which are most indicative nonlinear excitations. Note that for the case of anisotropic antiferromagnets with a uniaxial or rhombic magnetic anisotropy such traveling-wave solitons are present, they describe moving domain walls, see Refs. 14,15. It is interesting to sort whether the abovementioned absence of two specific nonlinear effects is an intrinsic property of an isotropic antiferromagnet or it appeared due to approximations done during transition to $\sigma-$model.

To answer this question it is necessary to proceed from a full system of equations for two vector variables $\mathbf{m}$ and $\mathbf{l}$, bound by the relation (2). Such an analysis, in princi-

ple, is considerably complicated as one has to deal with four nonlinear equations, rather than two angular variables for the unit vector $\mathbf{l}$ as for $\sigma-$model. However, we can limit our consideration to analysis of some concrete class of solutions in order to confirm the presence of solitons.

In this article, a class of solutions in a simple model of an antiferromagnet with consideration of only isotropic exchange interaction is pointed out. In such a solution the vector $\mathbf{m}$ is parallel to some direction and change its length only, while the vector $\mathbf{l}$ turns around it within some plane. It is appropriate to call these solutions as "planar". Within the class of such solutions, consistent description of properties of nonlinear waves and soliton dynamics is done.

The article is organized as following. In the Section 2 a model is formulated and effective equations of spin dynamics in terms of $\mathbf{m}$ and $\mathbf{l}$ without application of typical for $\sigma-$model approximations are presented, and the integrals motion are obtained. On the basis of these equations in the Section 3 the soliton structure is calculated for a one-dimensional case. The analysis of the dispersion law of solitons done in the Section 4 demonstrates these one-dimensional stable solitons are magnetic analogies of Lieb states known from one-dimensional Bose gas model.[19] Further in the Section 5 two-dimensional solitons describing magnetic vortices are analyzed.

II. MODEL, EFFECTIVE EQUATIONS OF SPIN DYNAMICS AND CONSERVATION LAWS.

Dynamical equations for the vectors $\mathbf{m}$ and $\mathbf{l}$ can be written as follows

$$\hbar S\frac{\partial \mathbf{m}}{\partial t} = \left(\mathbf{m}\times\frac{\delta W}{\delta \mathbf{m}}\right) + \left(\mathbf{l}\times\frac{\delta W}{\delta \mathbf{l}}\right),$$
$$\hbar S\frac{\partial \mathbf{l}}{\partial t} = \left(\mathbf{l}\times\frac{\delta W}{\delta \mathbf{m}}\right) + \left(\mathbf{m}\times\frac{\delta W}{\delta \mathbf{l}}\right), \quad (3)$$

where $W = W[\mathbf{m},\mathbf{l}] = \int w\{\mathbf{m},\mathbf{l}\}(d^dx/a^d)$ is the energy functional of an antiferromagnet, which is presented here for a magnet with a hypercubic lattice with dimension d, $w = w\{\mathbf{m},\mathbf{l}\}$ is the energy density, which depends on the vectors $\mathbf{m}$ and $\mathbf{l}$ and their spatial derivatives. In the standard expansion on gradients with account taken of Eq. (2), a general expression for w in the case of a purely isotropic antiferromagnet takes the form

$$w = JS^2\mathbf{m}^2 + \frac{1}{2}A_1a^2S^2\left(\nabla\mathbf{m}\right)^2 + \frac{1}{2}A_2a^2S^2\left(\nabla\mathbf{l}\right)^2 \ , \quad (4)$$

where J is the effective homogeneous exchange constant, the parameters A_1 and A_2 are determined by exchange integrals within one sublattice and between sublattices, respectively, S is an atomic spin and a is the lattice constant. For this energy, the magnetization vector $\mathbf{m}$ equals to zero in the ground state. It is worth noting, for the model (4) with the $A_1 = 0$ the $\sigma-$model representation is exact, while the values of both constants are important for the soliton solutions for antiferromagnet. Below, we will not specify relations between the constants A_1 and A_2 and their connections with some microscopic spin model.

The equations (3) have the obvious integral of motion, the whole system energy E, values of which coincide with the value of $W[\mathbf{m},\mathbf{l}]$ calculated for some concrete solution, and the field momentum $\mathbf{P}$, which would be described below. For an isotropic problem the total spin value $\mathbf{S}^{(tot)} = \int S\mathbf{m}d^dx/a^d$ is also an integral of motion. It can be derived from a dynamical equation for the spin density $\mathbf{m}$, with usage of energy form (4), that gives

$$\hbar\frac{\partial \mathbf{m}}{\partial t} = -\mathrm{div}\left[S\left(\mathbf{m}\times A_1\nabla\mathbf{m}\right) + S\left(\mathbf{l}\times A_2\nabla\mathbf{l}\right)\right]. \quad (5)$$

This expression determines the conservation law of the total spin $\mathbf{S}^{(tot)}$ in differential form. Its analysis allows also to point out a concrete exact class of solutions for the full set of equations (5). Let the vector $\mathbf{m}$ and its time and space derivatives at the initial moment of time are parallel to some direction, which can be chosen as the z axis. The equation (2) demonstrate that in this case the vector $\mathbf{l}$ and its derivatives lie in the perpendicular (x, y) plane. In virtue of (5) such geometry remains for subsequent moments of time, i.e. dynamical equations for an antiferromagnet allow a planar solution in the form of $\mathbf{m}||\mathbf{e}_z,\ \mathbf{l}\perp\mathbf{e}_z$. Accounting the constraint (2) the vectors $\mathbf{m}$ and $\mathbf{l}$ can be parameterized by two angular variables,

$$\mathbf{m} = \mathbf{e}_z\sin\mu,\ \ \mathbf{l} = \cos\mu\left(\mathbf{e}_x\cos\varphi + \mathbf{e}_y\sin\varphi\right), \quad (6)$$

where $\mathbf{e}_x$ and $\mathbf{e}_y$ are unit vectors directed along x and y axis, respectively. The initial isotropy of the problem in this case manifest itself in arbitrary directions of axis $\mathbf{e}_z$, $\mathbf{e}_x$ and $\mathbf{e}_y$, specific for the planar solution.

An important characteristic of the planar solution is that the system dynamics with new variablesμ, φ can be described by a simple Lagrangian

$$L = \int\frac{d^dx}{a^d}\left(-\hbar S\frac{\partial\varphi}{\partial t}\sin\mu - w\right), \quad (7)$$

where w is the energy density (4) presented through angular variables,

$$w = JS^2\sin^2\mu + \frac{1}{2}A_2a^2S^2\cos^2\mu\left(\nabla\varphi\right)^2 +$$
$$+\frac{1}{2}a^2S^2\left[A_1\cos^2\mu + A_2\sin^2\mu\right]\left(\nabla\mu\right)^2 \ . \quad (8)$$

Lagrangian approach allows one to obtain an expression for linear momentum of the magnetic excitation $\mathbf{P}$, which is a total field momentum of corresponding field,

$$\mathbf{P} = \hbar S\int\frac{d^dx}{a^d}(\nabla\varphi)\sin\mu \ . \quad (9)$$

The dynamical part of the Lagrangian (7) and the expression for momentum (9) contain singularities connected

with non differentiability of the azimuthal angle φ. This property of the variable φ plays a significant role in description of vortices dynamics in ferromagnets.[20] In our case, the presence of this singularity will also manifest itself essentially in description of solitons dynamics, either one-dimensional or two-dimensional, see Sections 4,5.

III. NON-LINEAR WAVES AND ONE-DIMENSIONAL SOLITONS.

Lets us consider dynamics of a simple magnetization wave propagating along some direction, say, the x-axis, with the velocity v. For such wave, $\mu = \mu(\xi)$, $\varphi = \varphi(\xi)$, $\xi = x - vt$. For an analysis of such solutions it is easier to start with the spin conservation equations (5), which can be integrated once and then gives an apparent relation of $\varphi' = d\varphi/d\xi$ (in this Section, the derivative over ξ is denoted by prime) and μ in the following form

$$\varphi' = \frac{\hbar v \sin\mu + C_1}{a^2 S A_2 \cos^2\mu} \tag{10}$$

where C_1 is an arbitrary constant. Using this expression it is possible to introduce the Lagrange equation $\delta L/\delta\mu$ in the form of the second order ordinary differential equation for $\mu\,(\xi)$. It is easy to demonstrate that this equation has the first integral, and for $\mu\,(\xi)$ one can obtain a simple equation with separating variables. Hence the problem allows a general analysis of nonlinear waves depending on one parameter, the wave velocity v, and containing, in a general case, two arbitrary constants C_1 and C_2. The explicit solution of this equation can be presented in elliptic functions.

First of all we are interested in soliton solutions for which, far from a soliton, at $\xi \to \pm\infty$, $\mu\,(\xi)$ turns zero, while $\varphi\,(\xi)$ has constant value. Therefore we consider only the case $C_1 = 0$. Then the equation for $\mu\,(\xi)$ in soliton solution acquires the following form

$$\begin{aligned} a^2\left[A_1\cos^2\mu + A_2\sin^2\mu\right](\mu')^2 = \\ = \sin^2\mu\left(2J - \frac{\hbar^2 v^2}{a^2 S^2 A_2\cos^2\mu}\right). \end{aligned} \tag{11}$$

Let us discuss properties of such soliton solutions. A simple analysis demonstrates that the soliton velocity has an upper limit, the value $c = 2\sqrt{JA_2}\hbar S/a$, which coincides with phase velocity of linear excitations (magnons) for antiferromagnet. This is a rather natural condition for traveling-wave solitons. It is worth noting that c does not depend on constant A_1, thus, it can be obtained in the framework of $\sigma-$model. However soliton states exist only at $A_1 > 0$. The latter is a formal confirmation of the fact that for their analysis one should go beyond this model.

The soliton solution of this equation can only be written through elliptic functions. The structure of the planar solitons in antiferromagnets, as well as the energy dependence on the soliton velocity, is quite common to that for solitons in spin nematic state.[21] Hence, we will not discuss it in details and limit ourselves with its qualitative analysis. First of all, the form of the solution depends highly on the value of soliton velocity v. If the velocity v is nearly c, the soliton amplitude $\mu_{\max}$ is small, proportional to $\sqrt{c - v}$. The maximal value of $\mu_{\max}$ is reached at the zero soliton velocity.

As follows from the equations (10) and (11), the values of φ at the right and left of the soliton differ by a certain value $\Delta\varphi$. In the case $A_1 = A_2$ the value of $\Delta\varphi = \pi$ and it is independent of the soliton velocity. For any other relation between A_1 and A_2, this limit value $\Delta\varphi = \pi$ appears at zero soliton velocity, but $\Delta\varphi < \pi$ for $v \neq 0$ and it vanishes at $v \to c$. In principle, all these features are common to that for a so-called rotary waves for easy plane ferromagnets, see for review Refs. 1,2, or the so-called dark solitons, which are well known in nonlinear optics.[22]

The energy of a soliton is one of most important soliton characteristics. Using Eqs. (10) and (11), the energy density w (8) can be easily present through the function $\mu(\xi)$ only. It is convenient to write down the soliton energy E as a definite integral over μ from $\mu = 0$ till the maximal value $\mu_{\max}$. Again, the explicit value of this integral can be written through a simple but long combination of elliptic integrals only. The exception is the limit case $A_1 = A_2$, for which the explicit form for soliton energy as a function of its velocity can be written as a simple square root dependence,

$$E = E_0\sqrt{1 - \frac{v^2}{c^2}} \tag{12}$$

where c is the spin wave speed, $E_0 = 2aS^2\sqrt{2JA}$ is the maximal soliton energy, corresponding to the zero soliton velocity $v = 0$, in the case $A = A_1 = A_2$.

IV. SEMICLASSICAL QUANTIZATION OF ONE-DIMENSIONAL SOLITONS.

The soliton energy E and momentum P are the for are the most natural soliton characteristics and the dependence $E(P)$ is the basis for their semiclassical quantization.[1,2] Within this approach, $E(P)$ dependence can be considered as a dispersion n law for quantum nonlinear elementary excitations that are described by solitons. Usually, this dependence, which is found from classical solutions, well reflects the properties of the corresponding quantum results.

As has been noted above, the energy is maximal for a stationary soliton with $v = 0$, and it vanishes at $v \to c$. The concrete dependence can be easily found by numerical estimates of corresponding integral, see Ref. 21. Concerning soliton momentum, the situation is not so easy. It is worth noting, the equation (10) gives $d\varphi/d\xi = 0$ at $v = 0$ and $C_1 = 0$, that formally means zero value

of momentum. On the other hand, for any $v \neq 0$ the soliton momentum $P(v)$ is finite, and the limit value of the function $P(v)$ at $v \to 0$ is also finite. For example, for simplest case $A_1 = A_2$ one can easily find $P = (\hbar S/a) \cdot \arccos(v/c)$, that gives $P \to \pm\pi\hbar S/2a$ at $v \to \pm 0$. Combining this dependence with Eq. (12), one can present the dispersion relation for this particular case as a periodic function,

$$E = E_0 \cdot \left|\sin\left(\frac{\pi P}{2P_0}\right)\right|, \; P_0 = \frac{2\pi\hbar S}{a} \,. \tag{13}$$

with universal period P_0. The question appears, whether or not these features, the periodicity of the dispersion relation and the value of period are model independent.

In principle, this problem can be overcame by detail investigation of the behavior of the soliton solution at small velocities, see Refs. 1,2 for more details. On the other hand, it is useful to present a general model-free discussion, as it has been done for domain walls in ferromagnets.[23,24] Let discuss this problem in more details; moreover, it will be useful for the description of dynamical properties of vortex-like two-dimensional solitons.

Indeed, according to Eq. (9), the soliton momentum contains a singularity related to the presence of the gradient of the azimuthal angle φ. Such singularity is an internal property of the Lagrangian, see Eq. (7). It becomes clear if we parameterized the spin variables of the planar solution through a three-dimensional vector $\mathbf{R}$, $\mathbf{R} = (X,\, Y,\, Z) = (m,\, l_x,\, l_y)$, whose components represent nontrivial variables for the planar solution, namely, a magnetization $m = m_z$ and two non-zero projections of the vector $\mathbf{l}$. Then the density of the dynamical part of the Lagrangian (7) can be written as

$$\mathbf{A}(\mathbf{R})\frac{\partial \mathbf{R}}{\partial t}, \; \mathbf{A}(\mathbf{R}) = \frac{\hbar S}{a} \cdot \frac{Z(Y\mathbf{e}_x - X\mathbf{e}_y)}{R(X^2+Y^2)}, \; R = |\mathbf{R}| \tag{14}$$

where the vector $\mathbf{A}$ has a singularity along the $Z-$axis. This Lagrangian coincides with that for a charged particle with the coordinate $\mathbf{R}$ in a magnetic field with the vector potential $\mathbf{A}$. This representation also holds true for a ferromagnet in terms of the Landau–Lifshitz equation; however, expressions for $\mathbf{A}$ in these two cases are different. We can readily show that, although the expressions for $\mathbf{A} = \mathbf{A}(\mathbf{R})$ are different for the cases of an antiferromagnetic planar solution and a ferromagnet, for Eq. (14) we have $\mathbf{B} = \mathrm{rot}\mathbf{A} = \hbar S\mathbf{R}/aR^3$ Thus, as in the case of a ferromagnet, Eq. (14) describes the vector potential of a magnetic monopole located at the origin. Therefore, the expressions for a momentum P of one-dimensional soliton can be obtained by the substitution $\partial\mathbf{R}/\partial t \to -v\partial\mathbf{R}\partial\xi$; it can be reduced to the same form as for a soliton in a ferromagnet by gauge transformation. We then can use the same method as in Refs. 23,24.

The formula for the one-dimensional soliton momentum $P = \int \mathbf{A}(\mathbf{R})d\mathbf{R}$, contains a singularity and is not invariant with respect to the gauge transformations of the vector potential $\mathbf{A}$. However, it is important that the vector $\mathbf{B}$ does not contain singularities on the sphere $\mathbf{R}^2 = 1$. Whence, it follows that the difference in the momenta of two different soliton states is a gauge-invariant quantity. Indeed, every soliton (e.g., solitons with different velocities) can be associated with a trajectory connecting certain points $\mathbf{R}^{(-)}$ and $\mathbf{R}^{(+)}$ lying in the equator of the sphere $\mathbf{R}^2 = 1$ (circle $Z = 0$ or $m = 0$). In this case, the momentum of this soliton is specified by the integral $\int \mathbf{A}d\mathbf{R}$ over this trajectory going from the point $\mathbf{R}^{(-)}$ to the point $\mathbf{R}^{(+)}$. Although different solitons (e.g., solitons with different velocities) have different values of the variable φ gat infinity, all of them have $m = 0$ at infinity; that is, they finish at the equator of the sphere $\mathbf{R}^2 = 1$. In this line, the integrand is exactly zero; therefore, the ends of the illustrating trajectories of two solitons that finish at different points in the great circle can be connected by a segment lying in this circle and can be considered to be closed. It is clear that the difference in the momenta of the two solitons is determined by the integral over the closed contour $\oint \mathbf{A}d\mathbf{R}$ bound by the trajectories describing these solitons. According to the Stokes theorem, this integral can be written as a flux of a vector $\mathbf{B} = \mathrm{rot}\mathbf{A}$ through the surface enclosed by this contour. Therefore, the difference in the momenta of two soliton states ΔP can be represented in the gauge invariant form

$$\Delta P \;= \frac{\hbar S}{a}\int \mathbf{B}d\mathbf{S} = \frac{\hbar S}{a}\int \cos\mu \; d\mu \; d\varphi \tag{15}$$

Here the variables $\pi/2 - \mu$ and φ can be considered as the standard spherical coordinates for the vector $\mathbf{R}$, and the integral is taken over the region on the sphere bound by the trajectories corresponding to these two solitons. It is natural to choose the equator as the line corresponding to $P = 0$, to which the soliton trajectories tend asymmetrically as the soliton amplitude decreases; this corresponds to $E \to 0$ and $v \to c$. The maximum soliton energy corresponds to a trajectory that passes through the "north pole" of the sphere; for this pole, we have $P = P_0/2$ and $E = E_{\max}$. The $V(P)$ and $E(P)$ dependencies are then qualitatively restored. Indeed, all trajectories corresponding to a soliton velocity in the range from $v = c$ to $v = 0$ or to a soliton momentum from zero to $P_0/2$ fill the gap between these two limit trajectories. Hence, the momentum increases continuously when going from the trajectory near the equator and when approaching the limiting trajectory with $\pm P_0/2$. As a soliton trajectory moves further in the second half of the upper hemisphere, the energy decreases and the momentum increases until this trajectory reaches the equator. Here, the energy is $E = 0$, the momentum (with allowance for the choice of its reference point) is determined by integral (15) over the entire upper hemisphere, and $P = P_0$.

Thus, as for domain walls in a ferromagnet,[23,24] a true periodic $E\,(P)$ dependence appears for a planar solitons in an antiferromagnets due to the topological properties of the Lagrangian. This fact should lead to specific fea-

tures in forced soliton motion, e.g., to oscillating soliton motion under the action of a constant force (Bloch oscillations) as was discussed in details by Kosevich in Ref. 25.

V. TWO-DIMENSIONAL SOLITONS - ANTIFERROMAGNETIC VORTICES WITH FERROMAGNETIC CORE

Let us consider the static and dynamic properties of two-dimensional topological solitons on the basis of the model given by Eq. (4). For two-dimensional planar solitons the Lagrange equation for the variable φ takes the form

$$\hbar S \frac{\partial \mu}{\partial t} \sin \mu = A_2 a^2 \nabla [\sin^2 \mu \, (\nabla \varphi)], \tag{16}$$

In the static case, according to this equation, a two-dimensional solution can be taken in the form

$$\varphi = m\chi + \varphi_0, \; \mu = \mu(r), \tag{17}$$

where r and χ are the polar coordinates in the plane of the system and φ_0 is an arbitrary angle. To have a continuous distribution of the vectors $\mathbf{m}$ and $\mathbf{l}$, the number m should be integer. The structure of the vortex core is determined by the function $\mu(r)$ for which the ordinary differential equation can be obtained

$$\begin{aligned} &\left[1 + \kappa^2 \sin^2 \mu\right] \cdot \left(\frac{d^2\mu}{dr^2} + \frac{1}{r}\frac{d\mu}{dr} \right) - \\ &- \sin\mu \cos\mu \left[\frac{1}{l_0^2} - \frac{A_2 m^2}{A_1 r^2} - \kappa^2 \cdot \left(\frac{d\mu}{dr}\right)^2 \right] = 0, \end{aligned} \tag{18}$$

$\kappa^2 = (A_2 - A_1)/A_1, \; l_0 = a\sqrt{A_1/2J}$ is the characteristic length scale. If the condition $A_1 = A_2 = A$ holds, Eq. (18) by substitution $\mu \to \pi/2 - \theta$ transforms into the equation describing the vortex in easy plane ferromagnet, see Refs. 1,2. It is easy to show that at $r \gg l_0$ the quantity μ reaches its equilibrium value $\mu = 0$, and the behavior near the coordinate origin is a power law: $\mu(r) - \pi/2 \propto r^m$. Such power dependence is characteristic of a out-of-plane vortex in ferromagnets. Thus, at the center of the planar antiferromagnetic vortex a nonsingular saturated core with approximately ferromagnetic order is formed, and in the vortex center the magnetization takes its maximal value, see Fig.1.

It is easy to show that the energy of a planar antiferromagnetic vortex, as well as of other topological defects, has a weak (logarithmic) divergence with an increase in the system size L, it can be written as

$$E = m^2 \frac{\pi A_2 S^2 a^2}{2} \cdot \ln\left(\frac{L}{\eta l_0}\right) \,, \tag{19}$$

where η is a numerical factor on the order of unity. Hence, the vortex with $m = \pm 1$ has the minimal energy, and further we will discuss only this case.

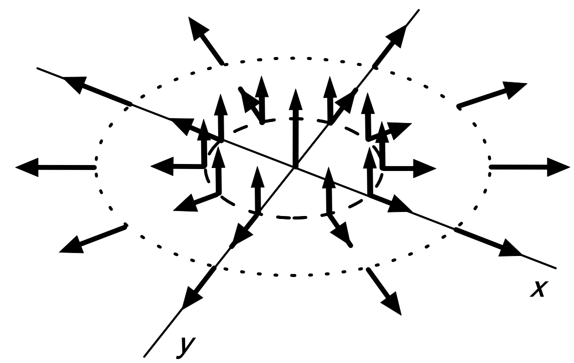

FIG. 1: Schematic distribution of the vector $\mathbf{l}$ (in-plane arrows with wide heads) and the vector $\mathbf{m}$ (vertical arrows) in the planar antiferromagnetic vortex with the vorticity $m = 1$. The core border, chosen as the line with $\mu = \pi/4$, is marked by the dashed line circle. The outermost circle (formally, the circle with $r \to \infty$, with the value of $\mu = 0$) is schematically shown by the dotted line circle.

It is interesting to compare the energy of this planar antiferromagnetic vortex with that for vortices in easy-plane antiferromagnets. In principle, planar antiferromagnetic vortices contains a ferromagnetic core with almost parallel sublattice magnetizations $\mathbf{m}_1$ and $\mathbf{m}_2$. On the first glance, this costs too much energy comparing with that for easy-plane antiferromagnetic vortices. But this energy difference enters the logarithmic multiplier, see Eq. (19). Thus, this difference is unimportant for many physical applications; for example, the only logarithmic dependence of the energy on the system size is manifesting the temperature of the Berezinskii-Kosterletz-Thouless transition in two-dimensional systems. Thus, both kinds of vortices can be important for a description of such transitions for real antiferromagnets.

Let us describe dynamic properties of the planar antiferromagnetic vortex, which are also nontrivial. In the framework of the $\sigma-$model, the solution describing any soliton freely moving with a velocity of $v < c$ can be obtained from the known immobile solution by the Lorentz transformation with the chosen speed c. However, the $\sigma-$model is inapplicable for the planar antiferromagnetic vortex considered above. Analysis shows that the motion of the planar antiferromagnetic vortex is possible only against the background of "spin flux," i.e., a nonzero value of $\nabla\varphi = \mathbf{k}$ at infinity. Vortex velocity $\mathbf{v}$ and $\mathbf{k}$ are related as $\hbar S \mathbf{v} = 2a^2 A_2 \cdot \mathbf{k}\cdot$; this relation can be derived using the same method as in Ref. 26 for a vortex in a ferromagnet. On the other words, far from the core of moving vortex the "condensate" is non-uniform, with $\nabla\varphi = \mathbf{k} \propto \mathbf{v} = d\mathbf{X}/dt$. Thus, the total energy of the system containing a freely moving planar antiferromagnetic vortex diverges as $\mathbf{v}^2 L^2$, L^2 is a system area, and the notion of the local inertial mass losses meaning. This property is known for vortices in ferromagnets or super-

fluid systems and corresponds to freezing of vortices in the condensate, see for review Ref. 4,15.

The problem of the forced motion of the planar antiferromagnetic vortex can be considered by analyzing the field momentum $\mathbf{P}$. Similar to a ferromagnet, Eq. (8) includes the non-differentiable expression, which leads to nontrivial features of the momentum of the topological soliton in these systems.[20] It is most simple to use the method proposed in Ref. 27 and to calculate the quantity $d\mathbf{P}/dt$ in the leading approximation in the vortex velocity $\mathbf{v}$. To this end, it is sufficient to use the immobile solution given by Eq. (10) with a change of $\mathbf{r}$ by $\tilde{\mathbf{r}}$, where $\tilde{\mathbf{r}} = \mathbf{r} - \mathbf{X}(t)$, $\mathbf{X} = \mathbf{X}(t) = X\mathbf{e}_x + Y\mathbf{e}_y$ is a coordinate of the vortex center. In this approximation, $\mu = \mu(\tilde{r})$, $\varphi = m\tilde{\chi}$, $\tilde{r} = |\tilde{\mathbf{r}}|$ and $\tilde{\chi} = \arctan[(y-Y)/(x-X)]$. Having in mind some general features of the vortex motion for the models with gyroscopic dynamics like in Lagrangian of Eq. (7), let us start with the general form of these term as in Eq. (14), not using the concrete form of the vector-potential $\mathbf{A}$.

In the leading approximation on the vortex velocity $\mathbf{v}$, the $\alpha-$th component of the time derivative of the vortex momentum, $d\mathbf{P}_0/dt$ with the taken into account the conditions $\partial\mathbf{R}/\partial t = -v_\alpha(\partial\mathbf{R}/\partial x_\alpha)$ can be rewritten as

$$\frac{dP_{0,\alpha}}{dt} = \int d^2x \cdot \frac{\partial R_i}{\partial x_\alpha}\frac{\partial R_j}{\partial x_\beta} v_\beta \left(\frac{\partial A_j}{\partial R_i} - \frac{\partial A_i}{\partial R_j}\right) \qquad (20)$$

As for the momentum of one-dimensional soliton, this expression contains gauge-invariant quantity $\mathbf{B} = \mathrm{rot}\mathbf{A}$, $\partial A_j/\partial R_i - \partial A_i/\partial R_j = \varepsilon_{ijk}(\mathrm{rot}\mathbf{A})_k$, instead of vector-potential $\mathbf{A}$ as itself. Then the direct calculation yields, $d\mathbf{P}_0/dt = G \cdot (\mathbf{e}_z \times \mathbf{V})$. Here the gyroconstant G, as well as the linear momentum for one-dimensional solitons (15), can be presented in the gauge invariant form $G = \hbar S \int \mathbf{B} d\mathbf{S}$, as a flux of the vector $\mathbf{B}$ through the area of the sphere $\mathbf{R}^2 = 1$, corresponding to the vortex, that gives $G = 2\pi\hbar S/a^2$.

VI. CONCLUSION.

Thus, beyond the $\sigma-$model approximation the isotropic antiferromagnets shows a reach variety of magnetic solitons with non-trivial static and especially dynamic properties. For one-dimensional magnet, soliton elementary excitations with a periodic dispersion law exists. These soliton excitations have common features with the so-called Lieb states,[19] which are well known in many condensed matter models. For two-dimensional case, planar antiferromagnetic vortices having non-singular macroscopic core with the saturated magnetic moment are found. The dynamic properties of these planar antiferromagnetic vortex are also unusual. Moving planar antiferromagnetic vortex is subjected to the gyroscopic force $G \cdot [\mathbf{e}_z, \mathbf{V}]$, equivalent to the Lorentz force for a charged particle in the uniform magnetic field, it is well known for vortices in easy-plane ferromagnets and superfluid systems, and is observed in experiments on the motion of magnetic bubbles and Bloch lines.[28] In contrast, gyroforce never appears in Lorentz-invariant $\sigma-$model equation; for a usual vortex in an antiferromagnet the gyroscopic force can be induced only by the strong external magnetic field and is absent for $H = 0$.[29] It is worth noting, both these non-trivial dynamical characteristics, period in dispersion law P_0 and gyroconstant G, can be written through gauge-invariant expressions of the common form. These quantities are independent on exchange integrals and depends only on a spin value S a single crystal parameter, namely, the interatomic distance a.

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